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IDENTIFICATION PROBLEMS FOR A STEADY-STATE MODEL OF MASS TRANSFER

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Coefficient identification problems for a steady-state mass-transfer model in the Oberbeck–Boussinesq approximation are considered. Optimality systems describing necessary conditions for the existence of an extremum are obtained, and, by analysis of their properties, conditions ensuring the uniqueness and stability of the solution are established.

Key words: *mass transfer, coefficient identification problem, uniqueness, stability.*

Introduction. In recent years, the need to determine effective mechanisms for controlling thermodynamic processes in viscous liquids has led to considerable attention being focused on optimal control problems for heat- and mass-transfer models. Theoretical investigation of the indicated problems has been the subject of a considerable number of papers (see, for example, [1–4]).

Along with optimal control problems, of great significance are identification problems for heat- and mass-transfer models, i.e., the determination (using additional information on the state of the medium) of the unknown densities of the boundary or distributed sources or the coefficients included in the differential equations or boundary conditions of the examined model. It should be noted that the solution of identification problems reduces to studying corresponding extremum problems with an adequate choice of the minimized quality functional. This allows control problems and identification problems to be investigated in terms of the theory of extremum problems of conditional optimization in Hilbert spaces.

Extremum problems of finding the source density have been studied in a number of papers (see, for example, [5–7]); less attention has been given to coefficient identification problems. We only note a paper [8], which, along with identification problems for source density, considers the problem of identification of the boundary condition coefficient for a thermal convection model.

1. Formulation of the Basic Boundary-Value Problem. The purpose of the present work is to study identification problems for the following mass transfer model:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f} + \beta_C C\boldsymbol{G}, \quad \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u}\Big|_\Gamma = \boldsymbol{g}; \tag{1.1}$$

$$-\lambda\Delta C + \boldsymbol{u}\cdot\operatorname{grad} C + kC = f \text{ in } \Omega, \quad C\Big|_{\Gamma_D} = \psi, \quad \lambda\Big(\frac{\partial C}{\partial n} + \alpha C\Big)\Big|_{\Gamma_N} = \chi. \tag{1.2}$$

Here Ω is a bounded region in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary Γ consisting of two parts Γ_D and Γ_N, $\boldsymbol{u}$ and C are the velocity and concentration of substance in the liquid, respectively, $p = P/\rho$, P is the pressure, $\rho = \text{const}$ is the density of the medium, $\nu > 0$ and $\lambda > 0$ are the kinematic viscosity and diffusion coefficient, which are constant, $\boldsymbol{f}$ and f are the volumetric density of the sources of mass forces and substance, $\boldsymbol{G} = -(0, 0, G)$ is the acceleration vector due to gravity, and β_C, $\boldsymbol{g}$, k, ψ, α, and χ are some functions. The quantities included in Eqs. (1.1) and (1.2), are dimensional, and the equations of the model are written in the SI system.

In [6, 7], the global solvability and local uniqueness of the boundary-value problem (1.1), (1.2) are proved and inverse extremum problems of finding the unknown densities of the mass and momentum sources are investigated.

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The present paper is concerned with a study of coefficient identification problems for the mass-transfer model considered, namely, with finding the unknown coefficients α and k, the density χ, and solutions $(\boldsymbol{u}, p, C)$ of problem (1.1), (1.2) from additional information on the desired solution. Difficulties in the investigation of identification problems is due to the fact that they are characterized by double nonlinearity — the nonlinearity of the model considered and the nonlinear inclusion (in the form of multipliers at C) of the unknown functions α and k in the model. Nevertheless, the structure of the differential equations of the mass-transfer model considered allows one to obtain two conditions on the initial data. The first condition is classical and ensures the uniqueness of the solution of the boundary-value problem (1.1), (1.2). The second condition is similar to the sufficient condition of the uniqueness of the solution of the coefficient identification problems for the linear convection–diffusion equation. Because the uniqueness conditions are rather awkward (because of the nonlinearity of the initial model), it is necessary to introduce analogs of the Reynolds, Rayleigh, and Prandtl numbers. Then, these conditions can be written in fairly simple and physically illustrative form.

As in [5], we use the spaces $H^s(D)$, $s \in \mathbb{R}$ and $L^r(D)$ or $\boldsymbol{H}^s(D)$ and $\boldsymbol{L}^r(D)$ for the vector functions, where D is a domain Ω (or its subset Q) or a boundary Γ (or its part Γ_N). We denote the scalar products in $L^2(\Omega)$, $L^2(Q)$, or $L^2(\Gamma_N)$ by $(\,\cdot\,,\,\cdot\,)$, $(\,\cdot\,,\,\cdot\,)_Q$, or $(\,\cdot\,,\,\cdot\,)_{\Gamma_N}$, respectively, the norm in $L^2(\Omega)$, $L^2(Q)$, or $L^2(\Gamma_N)$ by $\|\cdot\|$, $\|\cdot\|_Q$, or $\|\cdot\|_{\Gamma_N}$, the norm or seminorm in $H^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ by $\|\cdot\|_1$ or $|\cdot|_1$, the norm in $H^{1/2}(\Gamma_0)$ by $\|\cdot\|_{1/2,\Gamma_0}$, and the duality relation for the pair X and X^* by $\langle\,\cdot\,,\,\cdot\,\rangle_{X^*\times X}$ or by $\langle\,\cdot\,,\,\cdot\,\rangle$. Let the following conditions be satisfied:

1) Ω is a bounded domain in the space $\mathbb{R}^d$ with boundary $\Gamma \in C^{0,1}$ consisting of N coupled components $\Gamma^{(i)}$, $i = 1, 2, \ldots, N$;

2) $\Gamma_D \in C^{0,1}$, $\operatorname{meas}\Gamma_D > 0$, $\Gamma_N \in C^{0,1}$, $\Gamma_D \cap \Gamma_N = \varnothing$, and $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$.

We set $\boldsymbol{H}_0^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\colon\ \boldsymbol{v}|_\Gamma = 0\}$, $\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)\colon\ \operatorname{div}\boldsymbol{v} = 0\}$, $\tilde{\boldsymbol{H}}^1(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)\colon$ $\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_N} = 0,\ \int_{\Gamma^{(i)}} \boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma = 0,\ 1 \le i \le N\}$, $\tilde{\boldsymbol{H}}^{1/2}(\Gamma) = \{\boldsymbol{u}|_\Gamma\colon\ \boldsymbol{u} \in \tilde{\boldsymbol{H}}^1(\Omega)\}$, $Z = H^1(\Omega, \Gamma_D) \equiv \{S \in H^1(\Omega)\colon$ $S|_{\Gamma_D} = 0\}$, $L_0^2(\Omega) = \{p \in L^2(\Omega)\colon\ (p, 1) = 0\}$, and $L_+^2(D) = \{v \in L^2(D)\colon\ v \ge 0 \text{ on } D\}$.

We introduce bilinear and trilinear forms

$$a_0(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,d\Omega, \quad b(\boldsymbol{v}, q) = -\int_\Omega q\operatorname{div}\boldsymbol{v}\,d\Omega, \quad c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{v}\cdot\boldsymbol{w}\,d\Omega,$$

$$a_1(C, S) = \int_\Omega \nabla C\cdot\nabla S\,d\Omega, \quad c_1(\boldsymbol{u}, C, S) = \int_\Omega (\boldsymbol{u}\cdot\operatorname{grad} C)S\,d\Omega, \quad b_1(S, \boldsymbol{v}) = \int_\Omega \boldsymbol{b}S\cdot\boldsymbol{v}\,d\Omega$$

($\boldsymbol{b} = \beta_C\boldsymbol{G}$) which are continuous and have the following properties:

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -c(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad \forall\,\boldsymbol{u} \in \boldsymbol{V}, \qquad (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{H}^1(\Omega)\times\boldsymbol{H}_0^1(\Omega); \tag{1.3}$$

$$c_1(\boldsymbol{u}, C, S) = -c_1(\boldsymbol{u}, S, C) \quad \forall\,\boldsymbol{u} \in \boldsymbol{V}, \qquad (C, S) \in H^1(\Omega)\times Z; \tag{1.4}$$

$$a_0(\boldsymbol{v}, \boldsymbol{v}) \geqslant \delta_0\|\boldsymbol{v}\|_1^2 \quad \forall\,\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \qquad |c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \gamma_0\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1; \tag{1.5}$$

$$a_1(S, S) \geqslant \delta_1\|S\|_1^2 \quad \forall\,S \in Z, \qquad |c_1(\boldsymbol{u}, C, S)| \le \gamma_1\|\boldsymbol{u}\|_1\|C\|_1\|S\|_1; \tag{1.6}$$

$$|(\chi, S)_{\Gamma_N}| \leqslant \gamma_2\|\chi\|_{\Gamma_N}\|S\|_1, \quad |b_1(S, \boldsymbol{v})| \le \beta_1\|\boldsymbol{v}\|_1\|S\|_1, \quad |(C, S)_Q| \le \gamma_4\|C\|_Q\|S\|_1; \tag{1.7}$$

$$|(\alpha C, S)_{\Gamma_N}| \le \gamma_3\|\alpha\|_{\Gamma_N}\|C\|_1\|S\|_1, \qquad |(kC, h)| \le \gamma_5\|k\|\|C\|_1\|h\|_1. \tag{1.8}$$

Here δ_0, δ_1, γ_0, γ_1, $\ldots$, γ_5, and β_1 are constants which depend on Ω.

2. Formulation of the Identification Problem. Preliminary Results. We note that the boundary-value problem (1.1), (1.2) contains the parameters ν, β_C, λ, k, and α and the functions f, ψ,and χ, which describe the densities of the sources of substance (for example, impurity). To solve the boundary-value problem (1.1), (1.2), it is necessary to specify values of all parameters, boundary functions, and source densities.In practice, however, some of these parameters or densities are often unknown. In particular, the function k (the coefficient of decomposition of the substance due to chemical reactions) may be unknown. In this case, the solution $(\boldsymbol{u}, p, C)$ of problem (1.1), (1.2) should be sought together with the coefficient k using some information on the state of the medium. Information on

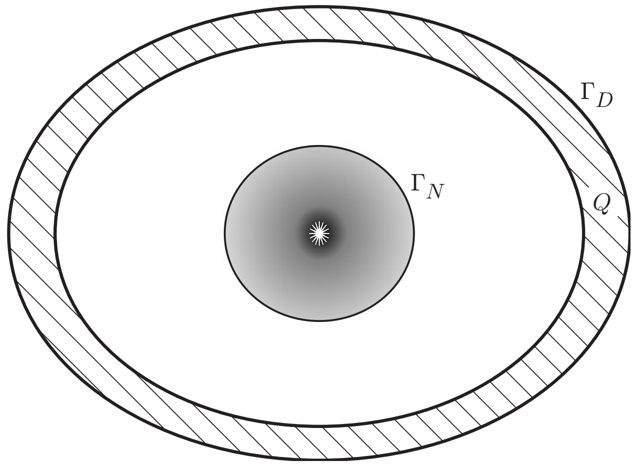

Fig. 1. Geometry of the flow domain.

the functions α and χ included in the boundary condition in (1.2) on the segment Γ_N of the boundary Γ may also be unknown. As an example of this situation, Fig. 1 shows a flow domain Ω which has the boundary Γ consisting of the external component $\Gamma_0 = \Gamma_D$ on which the Dirichlet condition is specified, and the internal component $\Gamma_1 = \Gamma_N$, through which the impurity enters the domain Ω. The concentration C on the outer boundary Γ_D and in some vicinity of it can be measured, but the inner boundary Γ_N may be inaccessible to measurements, and, hence, the quantities α and χ corresponding to Γ_N should be considered unknown. In this case, one needs to determine the quantities α and χ, together with the solution $(\boldsymbol{u}, p, C)$, from the measured concentration field C_d in the domain Q adjacent to the boundary Γ. A similar difficulty arises in problems of transboundary transfer of impurities.

From the aforesaid, we assume that the functions χ, α, and k included in system (1.1), (1.2) are unknown and are to be determined, together with the solution $(\boldsymbol{u}, p, C)$, from the minimum condition for some quality functional $\tilde{J}$. As $\tilde{J}$ we choose the functional $J_1(C) = \|C - C_d\|_Q^2$, where the function $C_d \in L^2(Q)$ models the concentration field or the functional $Q \subset \Omega$ for $J_2(C) = \|C - C_d\|_1^2$ measured in some subdomain $C_d \in H^1(Q)$.

We divide the set of initial data of problem (1.1), (1.2) into two groups: a group of controls, in which we include the functions χ, α, and k, and a group of fixed data, in which we include the unchangeable functions $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{g}$, f, and ψ. We assume that $u = (\chi, \alpha, k)$, $u_0 = (\boldsymbol{f}, \boldsymbol{b}, \boldsymbol{g}, f, \psi)$, and $\boldsymbol{x} = (\boldsymbol{u}, p, C)$ and that the control u can change on the set $K = K_1 \times K_2 \times K_3$. In this case, the following conditions are satisfied:

3) $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, $\boldsymbol{g} \in \tilde{\boldsymbol{H}}^{1/2}(\Gamma)$, and $f \in L^2(\Omega)$;

4) $K_1 \subset L^2(\Gamma_N)$, $K_2 \subset L^2_+(\Gamma_N)$, and $K_3 \subset L^2_+(\Omega)$ are nonempty closed convex sets.

Setting $X = \tilde{\boldsymbol{H}}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ and $Y = \boldsymbol{H}^{-1}(\Omega) \times L^2_0(\Omega) \times \tilde{\boldsymbol{H}}^{1/2}(\Gamma) \times Z^* \times H^{1/2}(\Gamma_D)$, we introduce the operator $F \equiv (F_1, F_2, F_3, F_4, F_5)$: $X \times K \to Y$, where $\langle F_1(\boldsymbol{x}, u), \boldsymbol{v}\rangle = \nu a_0(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b_1(C, \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle$, $F_2(\boldsymbol{x}, u) = \operatorname{div} \boldsymbol{u}$, $F_3(\boldsymbol{x}, u) = \boldsymbol{u}|_\Gamma - \boldsymbol{g}$, $\langle F_4(\boldsymbol{x}, u), S\rangle = \lambda a_1(C, S) + \lambda(\alpha C, S)_{\Gamma_N} + c_1(\boldsymbol{u}, C, S) + (kC, S) - (f, S) - (\chi, S)_{\Gamma_N}$, and $F_5(\boldsymbol{x}, u) = C|_{\Gamma_D} - \psi$. Multiplying the equations in (1.1) and (1.2) by the test functions and integrating

them, we obtain the problem in a weak formulation, consisting of finding a solution $\boldsymbol{x} \equiv (\boldsymbol{u}, p, C) \in X$ of the operator equation $F(\boldsymbol{x}, u) \equiv F(\boldsymbol{u}, p, C, \chi, \alpha, k) = 0$, which is equivalent to the relation

$$\nu a_0(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b_1(C, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega),$$

$$\lambda a_1(C, S) + \lambda(\alpha C, S)_{\Gamma_N} + c_1(\boldsymbol{u}, C, S) + (kC, S) = (f, S) + (\chi, S)_{\Gamma_N} \qquad \forall\, S \in Z, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \boldsymbol{u}|_\Gamma = \boldsymbol{g}, \qquad C|_{\Gamma_D} = \psi.$$

We examine the extremum problem

$$J(\boldsymbol{x}, u) \equiv \frac{\mu_0}{2}\tilde{J}(\boldsymbol{x}) + \frac{\mu_1}{2}\|\chi\|^2_{\Gamma_N} + \frac{\mu_2}{2}\|\alpha\|^2_{\Gamma_N} + \frac{\mu_3}{2}\|k\|^2 \to \inf,$$

$$F(\boldsymbol{x}, u) = 0, \qquad (\boldsymbol{x}, u) \equiv (\boldsymbol{u}, p, C, \chi, \alpha, k) \in X \times K, \tag{2.2}$$

where μ_l are nonnegative dimensional parameters. The values of these parameters allow one to adjust the relative contribution of each term in (2.2), and their dimensions allow one to match the dimensions of the quantities $\boldsymbol{u}$, p, and C of the main state to the dimensions of the quantities of the conjugate state. The parameters μ_1, μ_2, and μ_3 are also introduced to ensure the uniqueness of solutions of problem (2.2) (see Sec. 3). According to the general theory of extremum problems [9], we introduce into consideration an element $\boldsymbol{y}^* = (\xi, \sigma, \zeta, \eta, \tau) \in Y^*$, which will be called a conjugate state and a Lagrangian $\mathcal{L} : X \times K \times \mathbb{R}^+ \times Y^* \to \mathbb{R}$, where $\mathbb{R}^+ = \{x \in \mathbb{R}\colon\ x \geq 0\}$ using the formula

$$\mathcal{L}(\boldsymbol{x}, u, \lambda_0, \boldsymbol{y}^*) = \lambda_0 J(\boldsymbol{x}, u) + \langle \boldsymbol{y}^*, F(\boldsymbol{x}, u)\rangle \equiv \lambda_0 J(\boldsymbol{x}, u) + \langle F_1(\boldsymbol{x}, u), \xi\rangle$$

$$+ (F_2(\boldsymbol{x}, u), \sigma) + \langle \zeta, F_3(\boldsymbol{x}, u)\rangle_\Gamma + æ\langle F_4(\boldsymbol{x}, u), \eta\rangle + æ\langle \tau, F_5(\boldsymbol{x}, u)\rangle_{\Gamma_D}.$$

Here $\langle \zeta, \cdot\rangle_\Gamma = \langle \zeta, \cdot\rangle_{\tilde{\boldsymbol{H}}^{1/2}(\Gamma)^* \times \tilde{\boldsymbol{H}}^{1/2}(\Gamma)}$, $\langle \tau, \cdot\rangle_{\Gamma_D} \equiv \langle \tau, \cdot\rangle_{H^{1/2}(\Gamma_D)^* \times H^{1/2}(\Gamma_D)}$, æ is a parameter with dimension $[æ] = L_0^8/(T_0^2 M_0^2)$, L_0, T_0, and M_0 are characteristic quantities that have the dimensions of length, time, and mass, respectively, in meters, seconds, and kilograms (in the SI system). The indicated choice of [æ] ensures that the dimensions of the quantities ξ, σ, and η of the conjugate state coincide with the dimensions of the quantities $\boldsymbol{u}$, p, and C of the basic condition, i.e., the validity of the equalities $[\xi] = [\boldsymbol{u}] = L_0/T_0$, $[\eta] = [C] = M_0/L_0^3$, and $[\sigma] = [p] = L_0^2/T_0^2$. This allows ξ, σ, and η to be considered conjugate velocity, pressure, and concentration.

The following theorems are valid, which are proved similarly to the corresponding theorems in [6, 7].

Theorem 1. *Let conditions* 1–4 *be satisfied. Then, for any* $(\chi, \alpha, k) \in K$, *there exists at least one solution* $(\boldsymbol{u}, p, C) \in X$ *of problem* (1.1), (1.2) *and the following estimates are valid:* $\|\boldsymbol{u}\|_1 \leq M_{\boldsymbol{u}}(u_0, u)$, $\|p\| \leq M_p(u_0, u)$, *and* $\|C\|_1 \leq M_C(u_0, u)$. *Here* $M_{\boldsymbol{u}}$, M_p, *and* M_C *are nondecreasing continuous functions of the norms* $\|\boldsymbol{f}\|_{-1}$, $\|\boldsymbol{b}\|$, $\|\boldsymbol{g}\|_{1/2,\Gamma}$, $\|f\|$, $\|\psi\|_{1/2,\Gamma_D}$, $\|\chi\|_{\Gamma_N}$, $\|\alpha\|_{\Gamma_N}$, *and* $\|k\|$, *which tend to zero simultaneously with the norms* $\|\boldsymbol{f}\|_{-1}$, $\|\boldsymbol{g}\|_{1/2,\Gamma}$, $\|f\|$, $\|\psi\|_{1/2,\Gamma_D}$, *and* $\|\chi\|_{\Gamma_N}$. *If the quantities* $\boldsymbol{f}$, $\boldsymbol{g}$, f, ψ, χ, α, *and* k *are small* (*or the viscosity* ν *is large*) *so that the following condition is satisfied:*

$$\frac{\gamma_0}{\delta_0 \nu} M_{\boldsymbol{u}}(u_0, u) + \frac{1}{\delta_0 \nu} \frac{\beta_1 \gamma_1}{\delta_1 \lambda} M_C(u_0, u) < 1 \tag{2.3}$$

[*the constants* δ_i, γ_i, *and* β_1 *are introduced in* (1.5)–(1.8)], *the solution is unique.*

Theorem 2. *If conditions* 1–4 *are satisfied, let* $\mu_0 > 0$ *and* $\mu_l > 0$ *or* $\mu_0 > 0$ *and* $\mu_l \geq 0$ *and* K_l $(l = 1, 2, 3)$ *are bounded sets. Then, for* $\tilde{J} = J_k$ $(k = 1, 2)$ *there exists a solution of problem* (2.2).

Theorem 3. *If conditions* 1–4 *are satisfied, let the element* $(\hat{\boldsymbol{x}}, \hat{u}) \equiv (\hat{\boldsymbol{u}}, \hat{p}, \hat{C}, \hat{\chi}, \hat{\alpha}, \hat{k}) \in X \times K$ *be a local minimum point in problem* (2.2) *and let the functional* J *be continuously differentiable with respect to* $\boldsymbol{x}$ *at the point* $\hat{\boldsymbol{x}}$ *for any element* $u \in K$ *and be convex with respect to* u *for each point* $\boldsymbol{x} \in X$. *Then, there is a nonzero Lagrangian multiplier* $(\lambda_0, \boldsymbol{y}^*) \in \mathbb{R}^+ \times Y^*$ *such that the Euler–Lagrangian equation*

$$F'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{u})^* \boldsymbol{y}^* + \lambda_0 J'_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{u}) = 0$$

is valid and the minimum principle is satisfied:

$$\mathcal{L}(\hat{\boldsymbol{x}}, \hat{u}, \lambda_0, \boldsymbol{y}^*) \leq \mathcal{L}(\hat{\boldsymbol{x}}, u, \lambda_0, \boldsymbol{y}^*) \qquad \forall\, u \in K. \tag{2.4}$$

Theorem 4. *Let, for all* $u \in K$, *the conditions of theorem* 3 *and inequality* (2.3) *be satisfied. Then,* $\lambda_0 \neq 0$ *and the Lagrangian multiplier can be chosen to be equal to* $(1, \boldsymbol{y}^*)$.

We note that the Euler–Lagrangian equation is equivalent to the identities

$$\nu a_0(\boldsymbol{w}, \xi) + c(\hat{\boldsymbol{u}}, \boldsymbol{w}, \xi) + c(\boldsymbol{w}, \hat{\boldsymbol{u}}, \xi) + æ c_1(\boldsymbol{w}, \hat{C}, \eta) + b(\boldsymbol{w}, \sigma)$$

$$+ \langle \zeta, \boldsymbol{w}\rangle_\Gamma + \lambda_0 \langle J'_{\boldsymbol{u}}(\hat{\boldsymbol{x}}, \hat{u}), \boldsymbol{w}\rangle = 0 \qquad \forall\, \boldsymbol{w} \in \tilde{\boldsymbol{H}}^1(\Omega),$$

$$b(\xi, r) + \lambda_0 (J'_p(\hat{\boldsymbol{x}}, \hat{u}), r) = 0 \qquad \forall\, r \in L_0^2(\Omega), \tag{2.5}$$

$$æ[\lambda a_1(\varphi, \eta) + \lambda(\hat{\alpha}\varphi, \eta)_{\Gamma_N} + (\hat{k}\varphi, \eta) + c_1(\hat{\boldsymbol{u}}, \varphi, \eta) + \langle \tau, \varphi\rangle_{\Gamma_D}]$$

$$- b_1(\varphi, \xi) + \lambda_0 \langle J'_C(\hat{\boldsymbol{x}}, \hat{u}), \varphi\rangle = 0 \qquad \forall\, \varphi \in H^1(\Omega),$$

which, together with the ratio (2.1) and the minimum principle (2.4) form an optimality system describing the necessary extremum conditions for problem (2.2).

Theorems 1–4 establish sufficient conditions for the global solvability and local uniqueness of the initial boundary-value problem (1.1), (1.2), the solvability of the extremum problem (2.2), the validity of the Lagrangian principle, and the regularity of the Lagrangian multiplier. However, even in the case of satisfaction of condition (2.3), which ensures the uniqueness of the solution of problem (1.1), (1.2) and the regularity of the Lagrangian multiplier, these conditions does not imply the uniqueness of the solution of the extremum problem (2.2). To prove the uniqueness and stability of the solution of the extremum problem (2.2) with respect to perturbations of the function C_d, it is required to introduce more stringent constraints on the initial data.

3. Local Uniqueness and Stability of the Solution of the Identification Problem. Setting $\hat{M}_{\boldsymbol{u}} = \sup\limits_{u\in K} M_{\boldsymbol{u}}(u_0, u)$ and $\hat{M}_C = \sup_{u\in K} M_C(u_0, u)$, we introduce the parameters

$$\mathrm{Re} = \frac{\gamma_0 \hat{M}_{\boldsymbol{u}}}{\delta_0 \nu}, \quad \mathrm{R} = \frac{1}{\delta_0\nu}\,\frac{\beta_1\gamma_1}{\delta_1\lambda}\,\hat{M}_C, \quad \mathrm{Pr} = \frac{\delta_0\nu}{\delta_1\lambda}, \tag{3.1}$$

which are analogs of the dimensionless Reynolds Re, Rayleigh R and Prandtl Pr numbers, respectively, used in hydromechanics. We assume that

$$\mathrm{Re} + \mathrm{R} \equiv \frac{\gamma_0}{\delta_0\nu}\,\hat{M}_{\boldsymbol{u}} + \frac{1}{\delta_0\nu}\,\frac{\beta_1\gamma_1}{\delta_1\lambda}\,\hat{M}_C < \frac{1}{2}. \tag{3.2}$$

We note that the parameters introduced in (3.1) are dimensionless if the main norms $\|v\|$, $|v|_1$, and $\|v\|_1$ (v is any scalar quantity) are defined by the formulas

$$\|v\|^2 = \int\limits_\Omega v^2\,d\Omega, \quad |v|_1^2 = \int\limits_\Omega |\nabla v|^2\,d\Omega, \quad \|v\|_1^2 = l^{-2}\|v\|^2 + |v|_1^2.$$

Here $l = 1$ is a multiplier with the dimension $[l] = L_0$. Indeed, an analysis similar to the analysis performed in [8] shows that, in this case, the dimensions of the constants in (3.1) are defined by the relations $[\delta_i] = 1$, $[\gamma_i] = L_0^{1/2}$, $[\beta_1] = L_0^6/(T_0^2 M_0)$, $[\hat{M}_{\boldsymbol{u}}] = L_0^{3/2}/T_0$, and $[\hat{M}_C] = M_0/L_0^{5/2}$. This and the conditions $[\nu] = [\lambda] = L_0^2/T_0$ imply that the quantities Re, R, and Pr introduced in (3.1) are dimensionless.

To study the uniqueness and stability of the solution of problem (2.2) for $\tilde{J} = J_1$, we introduce two close functions $C_d^{(1)}$ and $C_d^{(2)} \in L^2(Q)$ and denote by $(\boldsymbol{x}_i, u_i, \boldsymbol{y}_i^*) \equiv (\boldsymbol{u}_i, p_i, C_i, \chi_i, \alpha_i, k_i, \xi_i, \sigma_i, \eta_i, \zeta_i, \tau_i)$ the solution of system (2.1), (2.4), (2.5) that correspond to $C_d^{(i)}$, in which it is necessary to set $\hat{\boldsymbol{x}} = \boldsymbol{x}_i$, $\hat{u} = u_i$, and $\boldsymbol{y}^* = \boldsymbol{y}_i^*$,

$$\lambda_0 = 1, \qquad \langle J_C'(\hat{\boldsymbol{x}}, \hat{u}), \varphi\rangle = \mu_0(C_i - C_d^{(i)}, \varphi)_Q, \qquad J_{\boldsymbol{u}}' = 0, \qquad J_p' = 0. \tag{3.3}$$

By virtue of Theorem 1 applied to $(\boldsymbol{u}_i, p_i, C_i)$ and relation (3.2), the following estimates are valid:

$$\|\boldsymbol{u}_i\|_1 \le \hat{M}_{\boldsymbol{u}}, \qquad \|C_i\|_1 \le \hat{M}_C, \qquad i = 1, 2; \tag{3.4}$$

$$\frac{\delta_0\nu}{2} < \delta_0\nu - \gamma_0\hat{M}_{\boldsymbol{u}} - \frac{\beta_1\gamma_1}{\delta_1\lambda}\,\hat{M}_C \le \delta_0\nu - \gamma_0\hat{M}_{\boldsymbol{u}} \le \delta_0\nu; \tag{3.5}$$

and by virtue of the condition $J_p' = 0$, we have $\operatorname{div}\xi_i = 0$, so $\xi_i \in \boldsymbol{V}$. We set $C_d = C_d^{(1)} - C_d^{(2)}$, $\chi = \chi_1 - \chi_2$, $\alpha = \alpha_1 - \alpha_2$, $k = k_1 - k_2$, $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ ($\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $p = p_1 - p_2$, and $C = C_1 - C_2$), $\xi = \xi_1 - \xi_2$, $\sigma = \sigma_1 - \sigma_2$, $\zeta = \zeta_1 - \zeta_2$, $\eta = \eta_1 - \eta_2$, and $\tau = \tau_1 - \tau_2$. In view of Theorem 4, the minimum principle (see Theorem 3) for the three parameters ($\boldsymbol{x}_i$, u_i, and $\boldsymbol{y}_i^*$) can be written as $\mathcal{L}(\boldsymbol{x}_i, u_i, 1, \boldsymbol{y}_i^*) \le \mathcal{L}(\boldsymbol{x}_i, u, 1, \boldsymbol{y}_i^*)$ for all $u \in K$. We note that the Lagrangian $\mathcal{L}$ is a continuously differentiable function of the controls χ, α, and k. Since K_1, K_2, and K_3 are convex sets, at the minimum point $u_i = (\chi_i, \alpha_i, k_i)$, the following conditions [10, p. 126] are satisfied:

$$\langle \mathcal{L}_\chi'(\boldsymbol{x}_i, u_i, 1, \boldsymbol{y}_i^*), \tilde{\chi} - \chi_i\rangle \equiv \mu_1(\chi_i, \tilde{\chi} - \chi_i)_{\Gamma_N} - æ(\tilde{\chi} - \chi_i, \eta_i)_{\Gamma_N} \ge 0 \quad \forall\,\tilde{\chi} \in K_1; \tag{3.6}$$

$$\langle \mathcal{L}_\alpha'(\boldsymbol{x}_i, u_i, 1, \boldsymbol{y}_i^*), \tilde{\alpha} - \alpha_i\rangle \equiv \mu_2(\alpha_i, \tilde{\alpha} - \alpha_i)_{\Gamma_N} + \lambda æ((\tilde{\alpha} - \alpha_i)C_i, \eta_i)_{\Gamma_N} \ge 0 \quad \forall\,\tilde{\alpha} \in K_2; \tag{3.7}$$

$$\langle \mathcal{L}_k'(\boldsymbol{x}_i, u_i, 1, \boldsymbol{y}_i^*), \tilde{k} - k_i\rangle \equiv \mu_3(k_i, \tilde{k} - k_i) + æ((\tilde{k} - k_i)C_i, \eta_i) \ge 0 \quad \forall\,\tilde{k} \in K_3. \tag{3.8}$$

Here, for example, $\mathcal{L}_\alpha'(\boldsymbol{x}_i, u_i, 1, \boldsymbol{y}_i^*)$ is the Gateaux derivative with respect to α at the point $(\boldsymbol{x}_i, u_i, 1, \boldsymbol{y}_i^*)$.

Setting $\tilde{k} = k_1$ for $i = 2$ and $\tilde{k} = k_2$ for $i = 1$ in inequality (3.8), we obtain

$$\mu_3(k_2, k) + \text{æ}(kC_2, \eta_2) \geq 0, \qquad -\mu_3(k_1, k) - \text{æ}(kC_1, k_1) \geq 0.$$

Combining these inequalities, we have

$$\mu_3\|k\|^2 \leq \text{æ}[(kC_2, \eta_2) - (kC_1, \eta_1)] = -\text{æ}[(kC_2, \eta) + (kC, \eta_1)]. \tag{3.9}$$

Similarly, from (3.6), (3.7) we obtain

$$\mu_1\|\chi\|^2_{\Gamma_N} \leq \text{æ}(\chi, \eta)_{\Gamma_N}, \qquad \mu_2\|\alpha\|^2_{\Gamma_N} \leq -\lambda\text{æ}[(\alpha C_2, \eta)_{\Gamma_N} + (\alpha C, \eta_1)_{\Gamma_N}]. \tag{3.10}$$

Combining relations (3.9) and (3.10), and taking into account (1.7), (1.8), and (3.4), we have

$$\mu_1\|\chi\|^2_{\Gamma_N} + \mu_2\|\alpha\|^2_{\Gamma_N} + \mu_3\|k\|^2$$
$$\leq -\text{æ}[\lambda(\alpha C_2, \eta)_{\Gamma_N} + \lambda(\alpha C, \eta_1)_{\Gamma_N} + (kC_2, \eta) + (kC, \eta_1) - (\chi, \eta)_{\Gamma_N}]$$
$$\leqslant \text{æ}[\|\eta\|_1(\gamma_2\|\chi\|_{\Gamma_N} + \gamma_3\lambda\hat{M}_C\|\alpha\|_{\Gamma_N} + \gamma_5\hat{M}_C\|k\|) + \|C\|_1\|\eta_1\|_1(\gamma_3\lambda\|\alpha\|_{\Gamma_N} + \gamma_5\|k\|)]. \tag{3.11}$$

For brevity, we introduce the notation

$$\|u\| \equiv \gamma_2\|\chi\|_{\Gamma_N} + \gamma_3\lambda\hat{M}_C\|\alpha\|_{\Gamma_N} + \gamma_5\hat{M}_C\|k\|; \tag{3.12}$$

$$\|z\| \equiv \gamma_1\|\eta_2\|_1\|\boldsymbol{u}\|_1 + \gamma_3\lambda\|\eta_2\|_1\|\alpha\|_{\Gamma_N} + \gamma_5\|\eta_2\|_1\|k\| + \mu_0\text{æ}^{-1}\gamma_4^2\|C\|_1. \tag{3.13}$$

Taking into account that, by virtue of (3.12), $\gamma_3\lambda\|\alpha\|_{\Gamma_N} + \gamma_5\|k\| \leq \|u\|/\hat{M}_C$, from (3.11) we have

$$\mu_1\|\chi\|^2_{\Gamma_N} + \mu_2\|\alpha\|^2_{\Gamma_N} + \mu_3\|k\|^2 \leq \text{æ}(\|\eta\|_1 + \|C\|_1\|\eta_1\|_1/\hat{M}_C)\|u\|. \tag{3.14}$$

Each solution $(\boldsymbol{u}_i, p_i, C_i, u_i)$ satisfies relation (2.1). Subtracting Eqs. (2.1) written for $\boldsymbol{u}_2, p_2, C_2, u_2$, from the corresponding equations for $\boldsymbol{u}_1, p_1, C_1, u_1$, we obtain

$$\begin{gathered}
\nu a_0(\boldsymbol{u}, \boldsymbol{v}) + [c(\boldsymbol{u}_1, \boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{v})] + b(\boldsymbol{v}, p) - b_1(C, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \\
\lambda a_1(C, S) + \lambda(\alpha_1 C, S)_{\Gamma_N} + c_1(\boldsymbol{u}_1, C, S) + (k_1 C, S) \\
= -c_1(\boldsymbol{u}, C_2, S) - \lambda(\alpha C_2, S)_{\Gamma_N} - (kC_2, S) + (\chi, S)_{\Gamma_N} \quad \forall\, S \in Z, \\
\operatorname{div} \boldsymbol{u} = 0, \qquad \boldsymbol{u}\Big|_{\Gamma} = 0, \qquad C|_{\Gamma_D} = 0.
\end{gathered} \tag{3.15}$$

Setting $\boldsymbol{v} = \boldsymbol{u} \in V$ in (3.15) and taking into account (1.3) and (1.4), we obtain

$$\nu a_0(\boldsymbol{u}, \boldsymbol{u}) = -c(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{u}) - b_1(C, \boldsymbol{u}); \tag{3.16}$$

$$\lambda a_1(C, C) + \lambda(\alpha_1 C, C)_{\Gamma_N} + (k_1 C, C) = -c_1(\boldsymbol{u}, C_2, C) - \lambda(\alpha C_2, C)_{\Gamma_N} - (kC_2, C) + (\chi, C)_{\Gamma_N}. \tag{3.17}$$

Using (1.5), (1.7), and (3.4), we have

$$|c(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{u})| \leq \gamma_0\|\boldsymbol{u}_2\|_1\|\boldsymbol{u}\|^2_1 \leq \gamma_0\hat{M}_{\boldsymbol{u}}\|\boldsymbol{u}\|^2_1, \qquad |b_1(C, \boldsymbol{u})| \leq \beta_1\|\boldsymbol{u}\|_1\|C\|_1.$$

In view of the last inequalities and (1.5), from (3.16), we obtain

$$\delta_0\nu\|\boldsymbol{u}\|^2_1 \leq \nu a_0(\boldsymbol{u}, \boldsymbol{u}) \leq \gamma_0\hat{M}_{\boldsymbol{u}}\|\boldsymbol{u}\|^2_1 + \beta_1\|C\|_1\|\boldsymbol{u}\|_1. \tag{3.18}$$

Relations (3.18) and (3.15) leads to

$$(\delta_0\nu/2)\|\boldsymbol{u}\|^2_1 \leqslant (\delta_0\nu - \gamma_0\hat{M}_{\boldsymbol{u}})\|\boldsymbol{u}\|^2_1 \leq \beta_1\|C\|_1\|\boldsymbol{u}\|_1.$$

From this, $\|\boldsymbol{u}\|_1 \leqslant (2\beta_1/(\delta_0\nu))\|C\|_1$. Using this estimate, relations (1.6)–(1.8), (3.4), and (3.12), and the conditions $\alpha_1 \geqslant 0$ and $k_1 \geqslant 0$, from (3.17), we derive

$$\delta_1\lambda\|C\|^2_1 \leq \lambda a_1(C, C) + \lambda(\alpha_1 C, C)_{\Gamma_N} + (k_1 C, C)$$
$$\leqslant (\gamma_1\hat{M}_C\|\boldsymbol{u}\|_1 + \gamma_2\|\chi\|_{\Gamma_N} + \gamma_3\lambda\hat{M}_C\|\alpha\|_{\Gamma_N} + \gamma_5\hat{M}_C\|k\|)\|C\|_1$$
$$\leq 2(\beta_1\gamma_1/(\delta_0\nu))\hat{M}_C\|C\|^2_1 + \|u\|\|C\|_1. \tag{3.19}$$

In view of (3.1) and (3.12), from (3.19) we obtain $\delta_1\lambda(1-2\,\mathrm{R})\|C\|_1 \le \|u\|\|C\|_1$. This and relation (3.1) lead to the following estimates for $\|C\|_1$ and $\|\boldsymbol{u}\|_1$:

$$\|C\|_1 \le \frac{\|u\|}{\delta_1\lambda(1-2\,\mathrm{R})}, \qquad \|\boldsymbol{u}\|_1 \leqslant \frac{2\beta_1}{\delta_0\nu}\frac{\|u\|}{\delta_1\lambda(1-2\,\mathrm{R})} = \frac{2\,\mathrm{R}\,\|u\|}{\gamma_1(1-2\,\mathrm{R})\hat{M}_C}. \tag{3.20}$$

For the Lagrangian multipliers ξ_i, σ_i, ζ_i, η_i, and τ_i for $\hat{u} = u_i$, $\hat{\boldsymbol{x}} = \boldsymbol{x}_i$, $\lambda_0 = 1$, and $\tilde{J} = J_1$, $i = 1, 2$ in identities (2.5) we set $\boldsymbol{w} = \xi_i$, $\varphi = \eta_i$, and $r = \sigma_i$. Then, in view of (1.3), (1.4) and (3.3), we obtain

$$\nu a_0(\xi_i, \xi_i) = -c(\xi_i, \boldsymbol{u}_i, \xi_i) - æ c_1(\xi_i, C_i, \eta_i); \tag{3.21}$$

$$æ[\lambda a_1(\eta_i, \eta_i) + \lambda(\alpha_i\eta_i, \eta_i)_{\Gamma_N} + (k_i\eta_i, \eta_i)] = b_1(\eta_i, \xi_i) - \mu_0(C_i - C_d^{(i)}, \eta_i)_Q. \tag{3.22}$$

Using (1.6)–(1.8) and (3.4), from (3.22) we have

$$\begin{aligned}\delta_1\lambda æ\|\eta_i\|_1^2 &\le \beta_1\|\xi_i\|_1\|\eta_i\|_1 + \mu_0(\|C_i\|_Q + \|C_d^{(i)}\|_Q)\|\eta_i\|_Q \\ &\leqslant [\beta_1\|\xi_i\|_1 + \mu_0\gamma_4(\gamma_4\|C_i\|_1 + \|C_d^{(i)}\|_Q)]\|\eta_i\|_1 \\ &\le [\beta_1\|\xi_i\|_1 + \mu_0\gamma_4(\gamma_4\hat{M}_C + \|C_d^{(i)}\|_Q)]\|\eta_i\|_1.\end{aligned}$$

This leads to the following estimate for η_i:

$$\|\eta_i\|_1 \leqslant \frac{\beta_1}{\delta_1\lambda æ}\|\xi_i\|_1 + \frac{\mu_0\tilde{M}_C}{\delta_1\lambda æ}, \qquad \tilde{M}_C = \gamma_4^2\hat{M}_C + \gamma_4\max(\|C_d^{(1)}\|_Q, \|C_d^{(2)}\|_Q). \tag{3.23}$$

In view of (1.5), (1.6), (3.4), and (3.23), we have

$$|c(\xi_i, \boldsymbol{u}_i, \xi_i)| \leqslant \gamma_0\hat{M}_{\boldsymbol{u}}\|\xi_i\|_1^2,$$

$$æ|c_1(\xi_i, C_i, \eta_i)| \le \gamma_1 æ\|\xi_i\|_1\|C_i\|_1\|\eta_i\|_1 \le (\beta_1\gamma_1/(\delta_1\lambda))\hat{M}_C\|\xi_i\|_1^2 + (\gamma_1\hat{M}_C/(\delta_1\lambda))\mu_0\tilde{M}_C\|\xi_i\|_1,$$

$$a_0(\xi_i, \xi_i) \ge \delta_0\|\xi_i\|_1^2.$$

Using these inequalities and (3.5), from (3.21) we derive

$$\frac{\delta_0\nu}{2}\|\xi_i\|_1^2 \leqslant \Big(\delta_0\nu - \gamma_0\hat{M}_{\boldsymbol{u}} - \frac{\beta_1\gamma_1}{\delta_1\lambda}\hat{M}_C\Big)\|\xi_i\|_1^2 \leqslant \frac{\gamma_1\hat{M}_C\mu_0\tilde{M}_C}{\delta_1\lambda}\|\xi_i\|_1. \tag{3.24}$$

Eqs. (3.24), (3.1), and (3.23) lead to the following estimates for ξ_i and η_i:

$$\|\xi_i\|_1 \leqslant \frac{2}{\delta_0\nu}\frac{\gamma_1\hat{M}_C\mu_0\tilde{M}_C}{\delta_1\lambda} = \frac{2\mu_0\tilde{M}_C\,\mathrm{R}}{\beta_1}, \qquad \|\eta_i\|_1 \leqslant \frac{\mu_0\tilde{M}_C(2\,\mathrm{R}+1)}{\delta_1\lambda æ}. \tag{3.25}$$

Next, we subtract Eqs. (2.5) from each other for $\lambda_0 = 1$ and $\tilde{J} = J_1$ written for $(\boldsymbol{x}_1, u_1, \boldsymbol{y}_1^*)$ and $(\boldsymbol{x}_2, u_2, \boldsymbol{y}_2^*)$. In view of (3.3), we have

$$\begin{aligned}&\nu a_0(\boldsymbol{w}, \xi) + c(\boldsymbol{u}_1, \boldsymbol{w}, \xi) + c(\boldsymbol{u}, \boldsymbol{w}, \xi_2) + c(\boldsymbol{w}, \boldsymbol{u}_1, \xi) + c(\boldsymbol{w}, \boldsymbol{u}, \xi_2) + æ c_1(\boldsymbol{w}, C_1, \eta) \\ &\quad + æ c_1(\boldsymbol{w}, C, \eta_2) + b(\boldsymbol{w}, \sigma) + \langle\zeta, \boldsymbol{w}\rangle_\Gamma = 0 \quad \forall\, \boldsymbol{w} \in \tilde{\boldsymbol{H}}^1(\Omega), \\ &b(\xi, r) = 0 \qquad \forall\, r \in L_0^2(\Omega), \\ &æ[\lambda a_1(\varphi, \eta) + \lambda(\alpha_1\tau, \eta)_{\Gamma_N} + \lambda(\alpha\tau, \eta_2)_{\Gamma_N} + (k_1\varphi, \eta) + (k\varphi, \eta_2) + c_1(\boldsymbol{u}_1, \varphi, \eta) \\ &\quad + c_1(\boldsymbol{u}, \varphi, \eta_2) + \langle\tau, \varphi\rangle_{\Gamma_D}] + b_1(\varphi, \xi) + \mu_0(C - C_d, \varphi)_Q = 0 \quad \forall\, \varphi \in H^1(\Omega).\end{aligned} \tag{3.26}$$

Setting $\boldsymbol{w} = \xi \in \boldsymbol{V}$, $\varphi = \eta \in Z$, and $r = \sigma \in L_0^2(\Omega)$ in (3.26) and using relations (1.3) and (1.4) and the conditions $\xi|_\Gamma = 0$ and $\eta|_{\Gamma_D} = 0$, we obtain

$$\nu a_0(\xi, \xi) + c(\xi, \boldsymbol{u}_1, \xi) = -c(\boldsymbol{u}, \xi, \xi_2) - c(\xi, \boldsymbol{u}, \xi_2) - æ c_1(\xi, C, \eta_2) - æ c_1(\xi, C_1, \eta); \tag{3.27}$$

$$\lambda a_1(\eta,\eta) + \lambda(\alpha_1\eta,\eta)_{\Gamma_N} + (k_1\eta,\eta)$$

$$= -c_1(\boldsymbol{u},\eta,\eta_2) - \lambda(\alpha\eta,\eta_2)_{\Gamma_N} - (k\eta,\eta_2) - æ^{-1}\mu_0(C - C_d,\eta)_Q - æ^{-1}b_1(\eta,\xi). \tag{3.28}$$

In view of the conditions $\eta \in Z$, $\alpha_1 \geqslant 0$, $k_1 \geqslant 0$, and (3.13), from (3.28) we derive

$$\delta_1\lambda\|\eta\|_1^2 \le (\gamma_1\|\eta_2\|_1\|\boldsymbol{u}\|_1 + \gamma_3\lambda\|\eta_2\|_1\|\alpha\|_{\Gamma_N} + \gamma_5\|\eta_2\|_1\|k\| + \mu_0 æ^{-1}\gamma_4^2\|C\|_1$$

$$+ \mu_0 æ^{-1}\gamma_4\|C_d\|_Q + \beta_1 æ^{-1}\|\xi\|_1)\|\eta\|_1 \equiv (\|z\| + \beta_1 æ^{-1}\|\xi\|_1 + \mu_0 æ^{-1}\gamma_4\|C_d\|_Q)\|\eta\|_1.$$

This leads to the estimate

$$\|\eta\|_1 \le \frac{\|z\|}{\delta_1\lambda} + \frac{\beta_1}{\delta_1\lambda æ}\,\|\xi\|_1 + \frac{\mu_0\gamma_4}{\delta_1\lambda æ}\,\|C_d\|_Q. \tag{3.29}$$

Reasoning as in the derivation of (3.24) and taking into account (3.29), from (3.27) we obtain

$$(\delta_0\nu - \gamma_0\hat{M}_{\boldsymbol{u}})\|\xi\|_1^2 \le 2\gamma_0\|\boldsymbol{u}\|_1\|\xi_2\|_1\|\xi\|_1 + \gamma_1 æ\|C\|_1\|\eta_2\|_1\|\xi\|_1$$

$$+ \frac{\gamma_1 æ\hat{M}_C}{\delta_1\lambda}\,\|z\|\|\xi\|_1 + \frac{\beta_1\gamma_1\hat{M}_C}{\delta_1\lambda}\,\|\xi\|_1^2 + \frac{\mu_0\gamma_1\gamma_4\hat{M}_C}{\delta_1\lambda}\,\|C_d\|_Q\|\xi\|_1.$$

After division by $\|\xi\|_1$, this relation and relation (3.5) imply that

$$\frac{\delta_0\nu}{2}\|\xi\|_1 \leqslant \Big(\delta_0\nu - \gamma_0\hat{M}_{\boldsymbol{u}} - \frac{\beta_1\gamma_1}{\delta_1\lambda}\,\hat{M}_C\Big)\,\|\xi\|_1$$

$$\le 2\gamma_0\|\boldsymbol{u}\|_1\|\xi_2\|_1 + \gamma_1 æ\|C\|_1\|\eta_2\|_1 + \frac{\gamma_1 æ\hat{M}_C}{\delta_1\lambda}\,\|z\| + \frac{\mu_0\gamma_1\gamma_4\hat{M}_C}{\delta_1\lambda}\,\|C_d\|_Q. \tag{3.30}$$

Using estimates (3.25) for $\|\xi_2\|_1$, and $\|\eta_2\|_1$, estimates (3.20) for $\|C\|_1$ and $\|\boldsymbol{u}\|_1$, and the relation $(\delta_1\lambda)^{-2} = \mathrm{Pr}/(\delta_0\nu\delta_1\lambda)$ implied by (3.1), we have

$$2\gamma_0\|\boldsymbol{u}\|_1\|\xi_2\|_1 + \gamma_1 æ\|C\|_1\|\eta_2\|_1 \le \gamma_0\,\frac{8\mu_0\tilde{M}_C\,\mathrm{R}^2}{\beta_1\gamma_1(1-2\,\mathrm{R})\hat{M}_C}\,\|u\| + \frac{\gamma_1\mu_0\tilde{M}_C(2\,\mathrm{R}+1)}{\delta_1\lambda\delta_1\lambda(1-2\,\mathrm{R})}\,\|u\|$$

$$\leqslant \frac{\mu_0\,\mathrm{R}\,\tilde{M}_C}{\beta_1(1-2\,\mathrm{R})\hat{M}_C}\Big(\frac{8\gamma_0\,\mathrm{R}}{\gamma_1} + \mathrm{Pr}\,(2\,\mathrm{R}+1)\Big)\|u\|. \tag{3.31}$$

Similarly, in view of (3.13) and the condition $\gamma_4^2\hat{M}_C \le \tilde{M}_C$, we obtain

$$\frac{\gamma_1 æ\hat{M}_C}{\delta_1\lambda}\,\|z\| \leqslant \frac{\gamma_1\hat{M}_C\gamma_1\mu_0\tilde{M}_C(2\,\mathrm{R}+1)2\,\mathrm{R}}{\delta_1\lambda\delta_1\lambda\gamma_1(1-2\,\mathrm{R})\hat{M}_C}\,\|u\|$$

$$+ \frac{\gamma_1\hat{M}_C}{\delta_1\lambda}\,\frac{\mu_0\gamma_4^2}{\delta_1\lambda(1-2\,\mathrm{R})}\,\|u\| + \frac{\gamma_1\hat{M}_C}{\delta_1\lambda}\,\frac{\mu_0\tilde{M}_C(2\,\mathrm{R}+1)\gamma_3\lambda}{\delta_1\lambda}\,\|\alpha\|_{\Gamma_N}$$

$$\leqslant \frac{\mu_0\tilde{M}_C\mathrm{Pr}\ \mathrm{R}}{\beta_1\hat{M}_C}\Big(\frac{2\,\mathrm{R}(2\,\mathrm{R}+1)}{1-2\,\mathrm{R}} + (2\,\mathrm{R}+1) + \frac{1}{1-2\,\mathrm{R}}\Big)\|u\| \le \frac{\mu_0\mathrm{Pr}\ \mathrm{R}(2\,\mathrm{R}+2)\tilde{M}_C}{\beta_1(1-2\,\mathrm{R})\hat{M}_C}\,\|u\|. \tag{3.32}$$

Taking into account (3.31) and (3.32), from (3.30) we have

$$\frac{\delta_0\nu}{2}\,\|\xi\|_1 \leqslant \frac{\mu_0\,\mathrm{R}\,\tilde{M}_C}{\beta_1(1-2\,\mathrm{R})\hat{M}_C}\Big(\frac{8\gamma_0\,\mathrm{R}}{\gamma_1} + \mathrm{Pr}\,(2\,\mathrm{R}+1) + \mathrm{Pr}\,(2\,\mathrm{R}+2)\Big)\|u\|$$

$$+ \frac{\mu_0\gamma_1\gamma_4}{\delta_1\lambda}\,\hat{M}_C\|C_d\|_Q = \frac{\mu_0 M_1\tilde{M}_C}{\beta_1(1-2\,\mathrm{R})\hat{M}_C}\,\|u\| + \frac{\mu_0\gamma_1\gamma_4}{\delta_1\lambda}\,\hat{M}_C\|C_d\|_Q.$$

Here $M_1 = M_1(\mathrm{R}, \mathrm{Pr}) = 8(\gamma_0/\gamma_1)\,\mathrm{R}^2 + \mathrm{Pr}\ \mathrm{R}(4\,\mathrm{R}+3)$ is a dimensionless constant dependent on the Rayleigh and Prandtl numbers. From this, it follows that

$$|\xi\|_1 \leqslant \frac{2\mu_0 M_1 \tilde{M}_C}{\delta_0 \nu \beta_1 (1-2\,\mathrm{R})\hat{M}_C}\,\|u\| + \frac{2\mu_0\gamma_4\,\mathrm{R}}{\beta_1}\,\|C_d\|_Q. \tag{3.33}$$

Using (3.33), we write (3.29) as

$$\|\eta\|_1 \leq \frac{1}{\delta_1\lambda}\,\|z\| + \frac{1}{\delta_1\lambda æ}\,\frac{2\mu_0 M_1 \tilde{M}_C}{\delta_0\nu(1-2\,\mathrm{R})\hat{M}_C}\,\|u\| + \frac{\mu_0\gamma_4}{\delta_1\lambda æ}(2\,\mathrm{R}+1)\|C_d\|_Q. \tag{3.34}$$

Relation (3.32) leads to

$$\frac{1}{\delta_1\lambda}\,\|z\| \leq \frac{\mu_0 \mathrm{Pr}\ \mathrm{R}(2\,\mathrm{R}+2)\tilde{M}_C}{\beta_1\gamma_1(1-2\,\mathrm{R})æ\hat{M}_C^2}\,\|u\|. \tag{3.35}$$

Relations (3.34) and (3.35) imply the following estimate for $\|\eta\|_1$ in terms of $\|u\|$ and $\|C_d\|_Q$:

$$\|\eta\|_1 \leq \frac{1}{æ}\Big(\frac{\mu_0 \mathrm{Pr}\ \mathrm{R}(2\,\mathrm{R}+2)\tilde{M}_C}{\beta_1\gamma_1(1-2\,\mathrm{R})\hat{M}_C^2} + \frac{\beta_1}{\delta_1\lambda}\,\frac{2\mu_0 M_1\tilde{M}_C}{\delta_0\nu\beta_1(1-2\,\mathrm{R})\hat{M}_C}\Big)\|u\| + \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)}{\delta_1\lambda æ}\,\|C_d\|_Q$$

$$= \frac{\mu_0}{æ}\,\frac{M_2\tilde{M}_C}{\beta_1\gamma_1(1-2\,\mathrm{R})\hat{M}_C^2}\,\|u\| + \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)}{\delta_1\lambda æ}\,\|C_d\|_Q \tag{3.36}$$

$[M_2 = 2\mathrm{Pr}\ \mathrm{R}(\mathrm{R}+1) + 2\,\mathrm{R}\,M_1]$.

Using (3.20), (3.25), and (3.36) to estimate the right side of inequality (3.14.), we obtain

$$æ\Big(\|\eta\|_1 + \frac{\|C\|_1\|\eta_1\|_1}{\hat{M}_C}\Big) \leq \Big(\frac{\mu_0 M_2\tilde{M}_C}{\beta_1\gamma_1(1-2\,\mathrm{R})\hat{M}_C^2} + \frac{\mu_0\tilde{M}_C(2\,\mathrm{R}+1)}{(\delta_1\lambda)^2(1-2\,\mathrm{R})\hat{M}_C}\Big)\|u\|$$

$$+ \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)}{\delta_1\lambda}\,\|C_d\|_Q = \frac{\mu_0 M_3\tilde{M}_C}{\beta_1\gamma_1\hat{M}_C^2}\,\|u\| + \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)}{\delta_1\lambda}\,\|C_d\|_Q. \tag{3.37}$$

Here $M_3 = M_3(\mathrm{R}, \mathrm{Pr})$ is a constant defined by the formula

$$M_3 = M/(1-2\,\mathrm{R}), \qquad M = \mathrm{R}(16\gamma_0\,\mathrm{R}^2/\gamma_1 + 8\mathrm{Pr}\ \mathrm{R}^2 + 10\mathrm{Pr}\ \mathrm{R} + 3\mathrm{Pr}). \tag{3.38}$$

Using (3.37) and the algebraic inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)\quad \forall\, a \in \mathbb{R}^+,\ b \in \mathbb{R}^+,\ c \in \mathbb{R}^+$, from (3.14) we have

$$\mu_1\|\chi\|^2_{\Gamma_N} + \mu_2\|\alpha\|^2_{\Gamma_N} + \mu_3\|k\|^2 \leqslant \frac{\mu_0 M_3\tilde{M}_C}{\beta_1\gamma_1\hat{M}_C^2}\,\|u\|^2 + \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)\|C_d\|_Q}{\delta_1\lambda}\,\|u\|$$

$$\leqslant \frac{3\mu_0 M_3\tilde{M}_C}{\beta_1\gamma_1\hat{M}_C^2}\,(\gamma_2^2\|\chi\|^2_{\Gamma_N} + \gamma_3^2\lambda^2\hat{M}_C^2\|\alpha\|^2_{\Gamma_N} + \gamma_5^2\hat{M}_C^2\|k\|^2) + \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)\|C_d\|_Q}{\delta_1\lambda}\,\|u\|.$$

The obtained inequality can be written as

$$\Big(\mu_1 - \frac{3\mu_0\gamma_2^2 M_3\tilde{M}_C}{\beta_1\gamma_1\hat{M}_C^2}\Big)\|\chi\|^2_{\Gamma_N} + \Big(\mu_2 - \frac{3\mu_0\gamma_3^2\lambda^2 M_3\tilde{M}_C}{\beta_1\gamma_1}\Big)\|\alpha\|^2_{\Gamma_N}$$

$$+ \Big(\mu_3 - \frac{3\mu_0\gamma_5^2 M_3\tilde{M}_C}{\beta_1\gamma_1}\Big)\|k\|^2 \leq \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)\|C_d\|_Q\|u\|}{\delta_1\lambda}. \tag{3.39}$$

We assume that the parameters μ_l included in (2.2) are such that

$$\mu_1 \geq \mu_0\,\frac{3\gamma_2^2 M_3\tilde{M}_C}{\beta_1\gamma_1\hat{M}_C^2} + 3\varepsilon\gamma_2^2, \qquad \mu_2 \geq \mu_0\,\frac{3\gamma_3^2\lambda^2 M_3\tilde{M}_C}{\beta_1\gamma_1} + 3\varepsilon\gamma_3^2\lambda^2\hat{M}_C^2,$$

$$\mu_3 \geq \mu_0\,\frac{3\gamma_5^2 M_3\tilde{M}_C}{\beta_1\gamma_1} + 3\varepsilon\gamma_5^2\hat{M}_C^2, \qquad \varepsilon = \mathrm{const} > 0. \tag{3.40}$$

From this and relation (3.20), we obtain

$$\|u\| = \|u_1 - u_2\| \le \frac{\mu_0\gamma_4(2\,\mathrm{R}+1)}{\varepsilon\delta_1\lambda}\,\|C_d^{(1)} - C_d^{(2)}\|_Q,$$

$$\|C_1 - C_2\|_1 \le \frac{\mu_0\gamma_4(1+2\,\mathrm{R})}{\varepsilon(\delta_1\lambda)^2(1-2\,\mathrm{R})}\,\|C_d^{(1)} - C_d^{(2)}\|_Q, \tag{3.41}$$

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\| \le \frac{2\mu_0\gamma_4\beta_1(1+2\,\mathrm{R})}{\varepsilon\delta_0\nu(\delta_1\lambda)^2(1-2\,\mathrm{R})}\,\|C_d^{(1)} - C_d^{(2)}\|_Q.$$

Relation (3.41) implies the uniqueness and stability of the solution of the identification problem (2.2) with respect to small perturbations of the specified function C_d in the norm $L^2(Q)$. We formulate the result obtained as the following theorem.

Theorem 5. *Let, in addition to conditions* 1–4, *the functions* $C_d^{(i)} \in L^2(Q)$ $(i = 1,2)$ *be specified and conditions* (3.2) *and* (3.40) *be satisfied, where* $\tilde{M}_C = \gamma_4^2\hat{M}_C + \gamma_4\max(\|C_d^{(1)}\|_Q, \|C_d^{(2)}\|_Q)$, *and the constant* M_3 *is determined in* (3.38). *We denote by* $((\boldsymbol{u}_i, p_i, C_i), u_i) \in X \times K$ *the solution of problem* (2.2) *that corresponds to* $C_d^{(i)}$. *Then, the stability estimates* (3.41) *are valid.*

Thus, in the present paper, the identification problem (2.2) was formulated and studied for a steady-state mass-transfer model. Two conditions on the initial data were obtained by analysis of the optimality system for problem (2.2) for $\tilde{J} = J_1$. These conditions ensure the uniqueness and stability of the solution of problem (2.2); the first condition has the form of the standard condition (3.2), which ensures the uniqueness of the solution of the initial boundary-value problem, and the second condition has the form of estimates (3.40) of the parameters μ_0, μ_1, μ_2, and μ_3 included in the extremum problem (2.2). Similar conditions can be obtained for $\tilde{J} = J$.

On the one hand, conditions (3.40) are similar to the uniqueness and stability conditions for the solution of the coefficient identification problems for the linear equation of transfer–convection–reaction (see, for example, [11]). On the other hand, these conditions contain compressed information on the initial nonlinear mass-transfer model in the form of the dimensionless constant M defined in (3.38). An analysis of the examined expression for M shows that, if in a study of problem (2.2), the dimensionless parameters Re, R, and Pr in (3.1) are not used, the expression for M is cumbersome. At the same time, from (3.38) it follows that M depends only on R and Pr and that $\mathrm{M} \to 0$ as $\mathrm{R} \to 0$. Hence, for fixed values of the parameters μ_l, relation (3.40) represent additional constraints on the Rayleigh number R, which, together with (3.2), ensures the uniqueness and stability of the solution of problem (2.2). We note that the constant M in (3.38) coincides with the constant in the similar uniqueness condition for the one-parameter identification problem for the thermal convection model considered in [8]. Hence, the constant M plays a key role in studies of the uniqueness of solutions of identification problems for steady-state heat and mass transfer models. We also note that, for fixed values of R and Pr, inequalities (3.40) imply that to ensure the uniqueness and stability of the solution of problem (2.2), the values of the parameters μ_1, μ_2, and μ_3 should be positive and exceed the constants on the right sides of inequalities (3.40).

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REFERENCES

1. M. D. Gunzburger, L. Hou, and T. P. Svobodny, “The approximation of boundary control problems for fluid flows with an application to control by heating and cooling,” *Comput. Fluids*, **22**, 239–251 (1993).
2. A. Capatina and R. Stavre, “A control problem in bioconvective flow,” *J. Math. Kyoto Univ.*, **37**, No. 4, 585–595 (1998).
3. H.-C. Lee and O. Yu. Imanuvilov, “Analysis of optimal control problems for the 2-D steady-state Boussinesq equations,” *J. Math. Anal. Appl.*, **242**, No. 2, 191–211 (2000).
4. G. V. Alekseev, “Solvability of steady-state boundary control problems for the thermal convection equations,” *Sib. Mat. Zh.*, **39**, No. 5, 982–998 (1998).

5. G. V. Alekseev, “Solvability of inverse extremum problems for the steady-state equations heat mass transfer,” *Sib. Mat. Zh.*, **42**, No. 5, 971–991 (2001).
6. G. V. Alekseev and E. A. Adomavichus, “Theoretical analysis of inverse extremal problems of admixture diffusion in viscous fluids,” *J. Inv. Ill-Posed Probl.*, **9**, No. 5, 435–468 (2001).
7. G. V. Alekseev, “Inverse extremum problems for the steady-state equations of mass transfer theory,” *Zh. Vychisl. Mat. Mat. Fiz.*, **42**, No. 3, 380–394 (2002).
8. G. V. Alekseev, “Inverse extremum problems for the steady-state equations of thermal convection,” *Vestnik NGU, Ser. Mat. Mekh., Informatika*, **6**, 7–33 (2006).
9. A. D. Yoffe and V. M. Tikhomirov, *Theory of Extremum Problems* [in Russian], Nauka, Moscow (1974).
10. J. Cea, *Optimisation. Theorie et Algorithms* [in French], Dunod, Paris (1971).
11. G. V. Alekseev and E. A. Kalinina, “Identification of the younger coefficient for the steady-state equation of convection–diffusion–reaction,” *Sib. Zh. Indust. Mat.*, **10**, No. 1, 3–16 (2007).